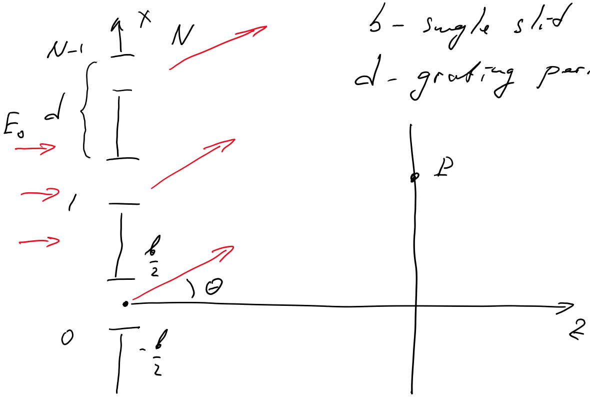
Fraunhofer diffraction on periodic structures

Let's first consider one-dimensional structures. That means our wave front will be modulated periodically along single coordinate. Let's introduce amplitude diffraction grating.



b - single slit widt

d - grating period.

Let's bring plane wave E_0 to such grating. The angle of incidence is normal. Our goal is to estimate field in point P.

$$I_P(\theta)$$

Our logic

$$I_p \to E_p \to E(k_x) \to E(x)$$

First, let's look at $E(x)$. In case of

normal incidence

$$E(x) = \begin{cases} E_0, & md - \frac{b}{2} \leq x \leq md + \frac{b}{2} \quad m = 0, 1, \dots, N-1 \\ 0, & \text{in remaining points} \end{cases}$$

Hence, we established the field at the slit $E(x)$

the rest is pure math

$$E(k_x) = \int_{-\infty}^{+\infty} E(x) e^{-ik_x x} dx = \sum_{m=0}^{N-1} \int_{md-\frac{b}{2}}^{md+\frac{b}{2}} E_0 e^{-ik_x x} dx =$$

$$= E_0 \sum_{m=0}^{N-1} \frac{-1}{ik_x} e^{-ik_x x} \Big|_{md-\frac{b}{2}}^{md+\frac{b}{2}} =$$

$$= E_0 b \left(\operatorname{sinc} \frac{k_x b}{2}\right) \underbrace{\sum_{m=0}^{N-1} e^{-ik_x md}}_{\Sigma \text{ geometric progression}}$$

$$\swarrow$$

$$q = e^{-ik_x d}$$

$$\Sigma = a_1 \frac{1-q^N}{1-q}$$

$$\Downarrow$$

$$E(k_x) = E_0 b \operatorname{sinc}\left(\frac{k_x b}{2}\right) \cdot \frac{1 - e^{-ik_x Nd}}{1 - e^{-ik_x d}}$$

We need to find

$$I \sim E(k_x)\, E^*(k_x)$$

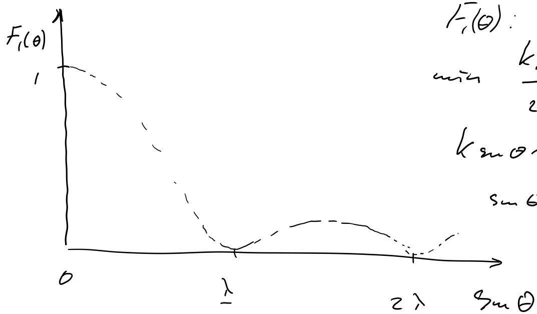
$$(1-e^{-ik_x d})(1-e^{ik_x d}) = 2(1-\cos k_x d) =$$

$$= 4\sin^2 k_x \frac{d}{2}$$

$$\Downarrow$$

$$\boxed{I(\theta) = I_0 \underbrace{\operatorname{sinc}^2 \frac{k_x b}{2}}_{F_1(\theta),\ \text{contribution of single slit}} \cdot \underbrace{\frac{\sin^2\left(\frac{k_x N d}{2}\right)}{\sin^2\left(\frac{k_x d}{2}\right)}}_{F_2(\theta)\ \text{contribution of multiple slits}}} \qquad k_x = k\sin\theta$$

I_0 is intensity of field when $\theta = 0$.



$F_1(\theta)$:

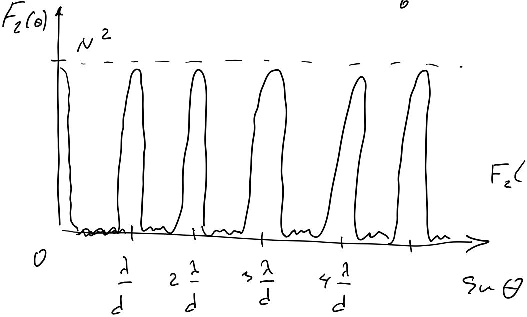
min $\frac{k_x b}{2} = \pi m$

$k \sin\theta\, b = 2\pi m$

$\sin\theta = m\frac{\lambda}{b}$

$m = \pm 1, \pm 2, \dots$

b $\bar{b}$



$F_2(\theta)_{max} = N^2$

Proof it l'Hôpital's rule

1) max $\frac{k_x d}{2} = m\pi$

$$\frac{2\pi}{\lambda}\sin\theta\,\frac{d}{2} = m\pi \qquad \boxed{d\sin\theta = m\lambda}$$

↳ main maximum

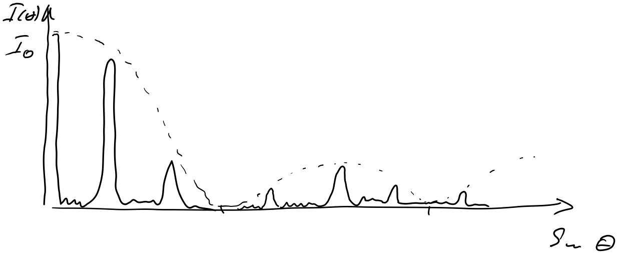
2) min $\frac{k_x N d}{2} = m\pi$ $\quad m \neq N, 2N, 3N\ldots$

$$\boxed{d\sin\theta = m\frac{\lambda}{N}}$$

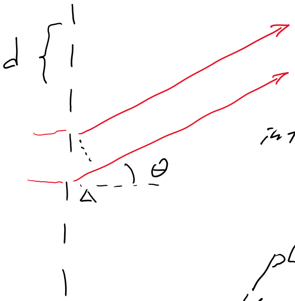
↳ additional minimum

Since N is very big, minima are very frequent.

Resulting function



Graphical interpretation

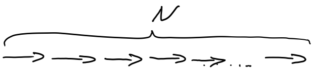


How main maximum appears? It appears thanks to collective interaction of slits.

$$\Delta = d \sin \Theta$$

phase difference between two neighboring slits

The maximum will appear when all oscillations are in phase.



$$\delta = k\Delta = 2\pi m$$

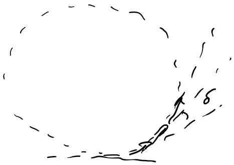
It is clear now, the Σ amplitude will be ~ N and intensity ~ N^2.

$$\delta = k\Delta = 2\pi m$$

$$\frac{2\pi}{\lambda} d \sin\theta = 2\pi m \Rightarrow \boxed{d\sin\theta = m\lambda}$$

Intensity is formed through interference of contributions of each slit.

Additional minimum appears when there is fixed phase difference.



The circle will close, when

$$\delta = \frac{2\pi m}{N}$$

$$\frac{2\pi}{\lambda} d\sin\theta = \frac{2\pi m}{N}$$

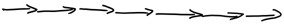
$$\Downarrow$$

$$\boxed{d\sin\theta = \frac{m\lambda}{N}}$$

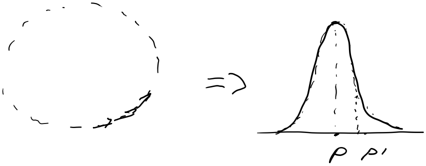
Now, let's look why main maxima are so narrow?

Let's take point P where main max appears.

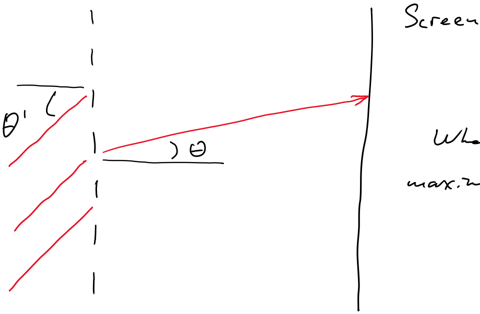
$\Delta = m\lambda \rightarrow P$



When we go slightly off to point P', we introduce small phase shift δ, hence the curve curls.

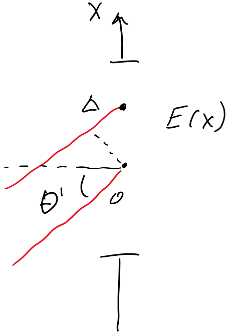


Light incidence at angle



$E(x) = ?$

Where are main maxima?

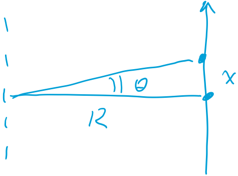


$$\delta = k\Delta = k x \sin\theta$$

$$E(x) = \begin{cases} E_0 e^{ikx\sin\theta'}, & \text{inside the slit} \\ 0, & \text{in remaining points} \end{cases}$$

$$\boxed{d(\sin\theta - \sin\theta') = m\lambda}$$

Demonstration



$$d \sin\theta = m\lambda$$

$d\theta = m\lambda$ for small θ

$$\theta = \frac{x}{R}$$

$$x \sim \frac{1}{d}$$